Ex-18

If G is a group such that $G/Z(G)$ is cyclic, where $Z(G)$ is centre of G, then show that G is abelian.

Soln:- For simplicity we write $Z(G) = N$.

Given G/N is cyclic.

Let gN be a generator of the cyclic group G/N, where $g \in G$.

Now we take two arbitrary elements x, y of G.

Therefore xN and yN are two elements of G/N

So there exist two integers m and n, such that $xN = (gN)^m$ and $yN = (gN)^n$

Now $xN = (gN)^m = (gN)(gN) \text{— } m \text{ times} = g^m N$

$\Rightarrow (g^m)^{-1}x \in N \Rightarrow g^{-m}x = n_1$; where $n_1 \in N$

$\Rightarrow x = g^m n_1$

Similarly,

we get $y = g^n n_2$, where $n_2 \in N$

Now $xy = (g^m n_1)(g^n n_2)$

$= g^m (n_1 g^n) n_2$

$= g^m (g^n n_1) n_2$ $[\because n_1 \in N = Z(G)]$

$= g^{m+n} n_1 n_2$ — (1)

Similarly, $yx = (g^n n_2)(g^m n_1)$

or $yx = g^n (n_2 g^m) n_1$

$= g^n (g^m n_2) n_1 \quad [\because n_2 \in M = Z(G)]$

$= g^{n+m} n_2 n_1$

$= g^{m+n} n_2 n_1 \quad \text{--- (2)}$

Since $n_1 n_2 = n_2 n_1$, so from (1) and (2), we get

$xy = yx$.

In other words G is abelian.

Ex-19 If N_1 and N_2 are two normal subgroups of a group G. Prove that $G/N_1 = G/N_2$ if and only if $N_1 = N_2$.

Sol^n If $N_1 = N_2$, then obviously $G/N_1 = G/N_2$.

Conversely

We assume that $G/N_1 = G/N_2$.

We have to show $N_1 = N_2$.

Clearly $N_1 \in G/N_1$.

$\therefore N_1 \in G/N_2$ (Since $G/N_1 = G/N_2$)

So N_1 is equal to some coset of N_2 in G.

We know that two cosets of N_2 in G are either identical or disjoint.

Since N_1 and N_2 both are subgroups of G, so at least the identity element e is common in N_1 and N_2, therefore N_1 and N_2 are not disjoint.

So we must have $N_1 = N_2$ (Since N_2 is also a member of G/N_2)

Ex-20

Let $Z(G)$ be the centre of the group G. If $a \in Z(G)$, then prove that the cyclic subgroup $\langle a \rangle$ of G generated by a is a normal subgroup of G.

Soln

Here $Z(G) = \{g \in G : gx = xg \ \forall x \in G\}$

If $a \in Z(G)$, then $ax = xa \ \forall x \in G$.

Let H be the cyclic subgroup generated by a.

ie $H = \langle a \rangle$

We have to show that H is a normal subgroup of G.

Let h be any element of H

Therefore $h = a^n$ for some integer n.

Let x be any element of G.

Now $xhx^{-1} = xa^nx^{-1} = (xax^{-1})^n$ $\left[\because (xax^{-1})(xax^{-1}) = xa^2x^{-1} \text{ and so on}\right]$

$= (axx^{-1})^n$ $[\because a \in Z(G)]$

$= (ae)^n$ [where e is the identity of G]

$= a^n \in H$

So $xhx^{-1} \in H \ \forall h \in H$ and $\forall x \in G$

Therefore H is a normal subgroup of G.

Hence if $a \in Z(G)$, then the cyclic subgroup generated by a is a normal subgroup of G.

Ex-21 Let a be any element of G. Show that the cyclic subgroup of G generated by a is a normal subgroup of the centralizer of a.

Solⁿ We know that the centralizer of a in a group G is $C(a) = \{x \in G : xa = ax\}$

Let H be the cyclic subgroup generated by a ie $H = \langle a \rangle$

We have to show that H is a normal subgroup of $C(a)$.

Let h be any element of H.

so $h = a^n$ for some integer n.

Now $a^n \cdot a = a^{n+1} = a \cdot a^n$

Therefore $h = a^n \in C(a)$

We know that $C(a)$ is a subgroup of G and H is also a subgroup of G.

Here $h \in H \Rightarrow h \in C(a)$

Therefore $H \subseteq C(a)$

So H is a subgroup of $C(a)$

Next we prove that H is a normal subgroup of $C(a)$.

Let x be any element of $C(a)$ and $h = a^n$ be any element of H.

We have $xhx^{-1} = xa^nx^{-1} = (xax^{-1})^n$

$= (axx^{-1})^n \quad [\because x \in C(a) \therefore ax = xa]$

$= (ae)^n \quad$ [where e is the identity of G]

$= a^n \in H$

So H is a normal subgroup of $C(a)$.

Ex-22

Let N be a normal subgroup of a group G. Show that $O(aN)/O(a)$ for any $a \in G$.

Sol^n

Let $O(a) = n$, so n is the least positive integer for which $a^n = e$, where e is the identity of the group G.

So we get $a^n N = eN = N$

$\Rightarrow (a \cdot a \cdots n \text{ times}) N = N$

$\Rightarrow aN \cdot aN \cdots n \text{ times} = N$

$\Rightarrow (aN)^n = N$,

Now $aN \in G/N$ and N is the identity of the quotient group G/N.

Therefore $O(aN)$ divides n.

In otherwords $O(aN) \,|\, O(a)$.

Ex-23

Let G be a non-abelian group of order pq, where p, q are primes then $O(Z(G)) = 1$

Sol^n

Given G is a non-abelian group.

So $G/Z(G)$ is not cyclic, where $Z(G)$ is the centre of G.

We know that $Z(G)$ is a subgroup of G.

So $O(Z(G))$ divides $O(G)$ ie pq

~~There possible order of~~

From Lagrange's theorem, we can conclude that the possible order of $Z(G)$ are $1, p, q$ and pq, since p, q are primes..

$O(Z(G))$ can not be pq, as G is not abelian and $Z(G)$ is abelian.

If $O(Z(G))$ is p.

Then $O\left(G/Z(G)\right) = \dfrac{O(G)}{O(Z(G))} = \dfrac{pq}{p} = q$, which is prime.

So $G/Z(G)$ is cyclic, which is a contradiction as G is non-abelian. So order of $Z(G)$ is not p.

By similar reason $O(Z(G))$ can not be q.

Therefore $O(Z(G)) = 1$.

Ex-24 If p is a prime number and G is a non-abelian group of order p^3, show that the centre of G has exactly p elements.

~~Proof~~

~~Let Z be the centre of G.~~

~~ie $Z = \{g \in G : gx = xg \ \forall \ x \in G\}$~~

~~Since the order of G is p^3, where p is a prime, so $G \neq \{e\}$, as Z is a subgroup of G.~~

Ex-25 Let $\phi : G \to G'$ be a group homomorphism. Prove that
CH-2019 $\text{Im}\,\phi$ is a subgroup of G'.

Sol^n $\text{Im}\,\phi = \phi(G) = \{\phi(x) : x \in G\}$

From definition of ϕ, $\text{Im}\,\phi$ is a subset of G'.

Since $\phi(e_G) = e_{G'}$, where e_G and $e_{G'}$ are the identity elements of G and G' respectively.

Therefore $e_{G'} \in \text{Im}\,\phi$

So $\text{Im}\,\phi$ is a non-empty subset of G'.

We take any two elements a', b' of $\phi(G)$, then there exist two elements a, b of G, such that $\phi(a) = a'$ and $\phi(b) = b'$.

Now $a'(b')^{-1} = \phi(a)\{\phi(b)\}^{-1} = \phi(a)\phi(b^{-1}) = \phi(ab^{-1})$ [Since ϕ is a homomorphism]

Since $ab^{-1} \in G$, therefore $\phi(ab^{-1}) \in \phi(G)$

Thus $a', b' \in \phi(G) \Rightarrow a'(b')^{-1} \in \phi(G)$

Therefore $\phi(G)$ is a subgroup of G'.

Ex-26

CH-2017 Prove that there does not exist an onto ~~homomo~~ homomorphism from the group $(\mathbb{Z}_6, +)$ to the group $(\mathbb{Z}_4, +)$.

Sol^n We first try to find all the homomorphisms from $\mathbb{Z}_6$ to $\mathbb{Z}_4$.

$(\mathbb{Z}_6, +)$ is a cyclic group generated by $[1]$.

We define a homomorphism f from $\mathbb{Z}_6$ to $\mathbb{Z}_4$.

Therefore $f([1])$ is a generator of the cyclic group $f(\mathbb{Z}_6)$.

For any $[a] \in \mathbb{Z}_6$, $f[a] = f([1]+[1]+\cdots a \text{ times})$
$= a f([1])$ [$\because f$ is a homomorphism].

Therefore the homomorphism f is completely known if $f([1])$ is known.

Now $O\{f([1])\}$ is a divisor of $O([1])$ ie 6.

Again $O\{f[1]\}$ is a divisor of $O(\mathbb{Z}_4)$ ie 4 as $f(\mathbb{Z}_6)$ is a subgroup of $\mathbb{Z}_4$.

Therefore $O\{f[1]\} = 1$ or 2.

So $f([1]) = [0]$ or $[2]$ [$\because O([1]) = O([3]) = 4$ in $\mathbb{Z}_4$]

If $f([1]) = [0]$, then $f([a]) = [0]$ $\forall [a] \in \mathbb{Z}_6$

In this case f is the trivial homomorphism.

So $f(\mathbb{Z}_6) = \{[0]\}$

If $f([1]) = [2]$, then $f([a]) = a[2]$

So we get $f([0]) = [0]$, $f([1]) = [2]$, $f([2]) = [0]$

$f([3]) = [2]$, $f([4]) = [0]$, $f([5]) = [2]$

Therefore in this case $f(Z_6) = \{[0], [2]\}$

In both these cases the image set $f(Z_6)$ is a proper subset of Z_4.

~~How~~ Therefore there does not exist an onto homomorphism from the group $(Z_6, +)$ to the group $(Z_4, +)$

Ex-27 Show that the mapping $f : (C', o) \to (R', *)$ defined

CH-2008

by $f(z) = |z|$ for all $z \in C'$, where (C', o) denotes the multiplicative group of non-zero complex numbers and $(R', *)$ denotes the multiplicative group of non-zero real numbers is a homomorphism and find Ker f.

Soln The mapping $f : (C', o) \to (R', *)$ is defined by $f(z) = |z|$ for all $z \in C'$.

~~Now~~ We take any two non-zero complex numbers z_1 and z_2 of C'.

~~Then~~ ~~f~~

Now $f(z_1 o z_2) = |z_1 z_2| = |z_1||z_2| = f(z_1) * f(z_2)$.

Therefore $f(z_1 o z_2) = f(z_1) * f(z_2)$ for all $z_1, z_2 \in C'$

~~The~~ So the mapping f is a homomorphism.

1 is the identity element of R'

Therefore Ker $f = \{z \in C' : f(z) = 1\} = \{z \in C' : |z| = 1\}$

Ex-28 If f is a homomorphism of G onto G' and g a homomorphism of G' onto G'', show that gof is a homomorphism of G onto G''. Also show that the kernel of f is a subgroup of that of gof.

Soln Since f is a mapping of G onto G' and g is a mapping of G' onto G'', so the composite mapping gof is a mapping of G onto G''.

We have $gof(x) = g\{f(x)\}\ \forall\ x \in G$.

Next we take any two elements a, b of G.

Now $(gof)(ab) = g\{f(ab)\}$

$= g\{f(a)f(b)\}$ [$\because$ f is a homomorphism]

$= g\{f(a)\}\, g\{f(b)\}$ [$\because$ g is a homomorphism]

$= \{(gof)(a)\}\{(gof)(b)\}$

Therefore (gof) is a homomorphism of G onto G'.

Let K be the kernel of gof.

So $K = \{x \in G : (gof)(x) = e''$, where e'' is the identity of $G''\}$

Again let K' be the kernel of f.

So $K' = \{x \in G : f(x) = e'$, where e' is the identity of $G'\}$

We know that K and K' are the normal subgroups of G. Therefore to show that K' is a subgroup of K it is sufficient to prove that $K' \subseteq K$.

We take $k' \in K'$.

Therefore $f(k') = e'$.

Now $g \circ f(k') = g\{f(k')\} = g(e') = e''$.

which implies $k' \in K$.

Thus $K' \subseteq K$

So the kernel of f is a subgroup of the kernel of $g \circ f$.

Ex-29 If n be any given positive integer, show that the mapping $f: C_0 \to C_0$ defined by $f(z) = z^n$, where C_0 is the multiplicative group of non-zero complex numbers, is an endomorphism. What is the kernel of this endomorphism?

Soln First we take any two non-zero complex numbers z_1 and z_2 and obviously they belong to C_0.

Therefore $f(z_1) = z_1^n$ and $f(z_2) = z_2^n$.

Now $f(z_1 z_2) = (z_1 z_2)^n = z_1^n z_2^n$ [$\because$ multiplication is commutative in C_0]

$= f(z_1) f(z_2)$

i.e. we have $f(z_1 z_2) = f(z_1) f(z_2)$

Therefore f is an endomorphism of C_0.

The identity of C_0 is 1.

Let kernel of f be K.

$\therefore K = \{z \in C_0 \mid f(z) = 1\}$

Now $f(z) = 1 \Rightarrow z^n = 1 \Rightarrow z = (1)^{1/n}$

Therefore z is an nth root of unity.

So kenel of f consists of the n nth roots of unity.

Note:– Endomorphism :- A homomorphism from a group G to itself is called an endomorphism of G.

Ex-30 (CT-2024) Let $(G, *)$ be a group and the mapping $f: G \to G$ be defined by $f(g) = g^{-1}$, $g \in G$. Show that f is an isomorphism if and only if G is abelian.

Soln We first assume that G is abelian..

~~Here f is~~

Here the mapping $f: G \to G$ be defined by $f(g) = g^{-1}$, $g \in G$.

We take any two elements g_1, g_2 of G.

Now $f(g_1 * g_2) = (g_1 * g_2)^{-1} = (g_2 * g_1)^{-1}$ [$\because$ G is abelian]

$$= g_1^{-1} * g_2^{-1} = f(g_1) * f(g_2)$$

Therefore f is an homomorphism of G.

Again for two elements g_1, g_2 of G.

$$f(g_1) = f(g_2)$$

$$\Rightarrow g_1^{-1} = g_2^{-1}$$

$$\Rightarrow (g_1^{-1})^{-1} = (g_2^{-1})^{-1}$$

$$\Rightarrow g_1 = g_2$$

So f is one-one.

Again for any $g \in G$ (codomain), there exists an element $g^{-1} \in G$ (domain) such that $f(g^{-1}) = (g^{-1})^{-1}$
$= g$.

Therefore g is onto.

Hence f is an isomorphism of G.

ie f is an automorphism of G.

Conversely

Let f be an isomorphism of G.

~~Now f(a*b)~~

We take any two elements g_1, g_2 of G.

Now $f(g_1 * g_2) = (g_1 * g_2)^{-1} = g_2^{-1} * g_1^{-1} = f(g_2 * g_1)$ [$\because$ f is an isomorphism]

Therefore $f(g_1 * g_2) = f(g_2 * g_1)$

$\Rightarrow g_1 * g_2 = g_2 * g_1$ [$\because$ f is an isomorphism, so f is one-one]

Hence G is an abelian group.

Ex-31
CU-2017

Let G be a commutative group of order n. If $\gcd(m,n)=1$, prove that the mapping $\phi : G \to G$ defined by $\phi(x) = x^m$, $x \in G$ is an isomorphism.

Solⁿ For any $a, b \in G$.

~~$\phi(ab) = (ab)^m$~~

$\phi(ab) = (ab)^m = a^m b^m$ [Since G is commutative]

$= \phi(a)\phi(b)$

So $\phi(ab) = \phi(a)\phi(b)$.

Therefore ϕ is a homomorphism.

Again $p \in \operatorname{Ker}\phi$

$\Rightarrow \phi(p) = e$ (the identity element of G)

$\Rightarrow p^m = e$

$\Rightarrow O(p) \mid m$.

Again $p \in G$ $\therefore$ $O(p) \mid n$ [$\because O(G) = n$]

Therefore $O(p) \mid \gcd(m,n)$

$\Rightarrow O(p) \mid 1$ [Given $\gcd(m,n)=1$]

$\Rightarrow p = e$

Hence $\operatorname{Ker}\phi = \{e\}$

So ϕ is one-one.

Again G is finite and ϕ is one-one, therefore ϕ is onto.

Therefore ϕ becomes an isomorphism.

Ex-32
CH-2021
Let $GL_n(R)$ be the general linear group over R and $SL_n(R)$ be the special linear group over R. Prove that $GL_n(R)/SL_n(R) \cong R^*$, where R^* is the group under usual multiplication of non-zero real numbers.

Sol^n

Here $GL_n(R)$ is the set of all $n \times n$ non-singular matrices over R.

$\therefore$ And $SL_n(R)$ is the set of all $n \times n$ matrices over R, whose determinant is 1.

We have to show $GL_n(R)/SL_n(R) \cong R^*$.

We first define a mapping $f: GL_n(R) \to R^*$ as

$f(A) = |A|$ [where $|A|$ = determinant of A]

Clearly f is onto.

Let A, B be any two elements of $GL_n(R)$,

then $f(AB) = |AB| = |A||B| = f(A) f(B)$

So f is an onto homomorphism.

Next we try to find $Ker f$.

The identity element of R^* is 1.

Therefore $Ker f = \{A : A \in GL_n(R) \text{ and } f(A) = 1\}$
$= SL_n(R)$

Hence by 1st isomorphism theorem, we get

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$$GL_n(R)/SL_n(R) \cong R^*.$$